Children's Intuitive Models of Multiplicaton and Division

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60 female students were observed four times during Grades 2 and 3 as they solved the same set of 24 multiplication and division problems with a wide variety of semantic structures. Students used three main intuitive models for both multiplication and division problems: direct counting, repeated addition and multiplication operations with a fourth model, repeated subtraction occurring only in division problems. The most popular model was repeated addition. Children's intuitive understanding of multiplication and division developed largely as a result of their recognising the equal group structure common to all multiplicative structures. The findings are in contrast to those of Fischbein et al. (1985).

Several recent studies have shown that children can solve a variety of multiplicative problems long before being instructed on the operations of multiplication and division. Kouba (1989) found that 30% of Grade 1 and 70% of Grade 2 children could solve simple equivalent set problems. Mulligan (1992) found a steady increase in success rate on similar problems from over 50% at the beginning of Grade 2 to nearly 95% at the end of Grade 3. More recently Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) found that even kindergarten children could learn to solve multiplicative problems.

When solving multiplication and division word problems children use a range of solution strategies, and from this it has been inferred that they have acquired various intuitive models of multiplication and division (Fischbein, Deri, Neri & Merino, 1985; Kouba, 1989; Greer, 1992; Mulligan, 1991). The interest in intuitive models lies in the proposition that they are formed early on in elementary contexts and can strongly influence students' understanding of more complex multiplicative situations in secondary school and adulthood, often negatively (Fischbein et al, 1985; Graeber & Tirosh, 1989; Simon, 1993). However, it is not yet clear exactly what an intuitive model is, how intuitive models are related to the semantic structure of the problems to be solved, and how they develop over time. The present paper attempts to throw light on these questions using data from a longitudinal study of children in Grades 2 and 3.

Background

One-step multiplicative word problems can be classified according to the nature of the quantities involved and the relation between them (Nesher, 1988; Vergnaud, 1988). Greer (1992) lists four categories which primarily apply to problems involving whole numbers:

- equivalent sets (e.g. 2 tables, each with 4 children)
- multiplicative comparison (e.g. 3 times as many boys as girls)
- cartesian product (e.g. the number of possible boy-girls pairs)
- rectangular arrays (e.g. 3 rows of 4 children),

and a further six categories which readily admit fractions and decimals. It has been found that mathematically equivalent problems of different semantic structure evoke different solution strategies and vary widely in difficulty (Bell, Fischbein, & Greer, 1984; Bell, Greer, Grimison, & Mangan, 1989; Brown, 1981, De Corte, Verschaffel, & Van

Coillie, 1988; Nesher, 1988; Vergnaud, 1988).

Classification of semantic structure is clearly somewhat arbitrary, in that the categories can be extended, collapsed or refined, depending on the purpose of the investigation. For example, Kouba (1989) proposed two categories of equivalent groups division problems based on the physical nature of the objects involved. The crucial point is that the semantic structure of a problem is determined by a researcher prior to its presentation to children and does not necessarily indicate how children will solve the problem.

Several studies of the actual strategies children use to solve multiplicative problems have been classified in two ways: 'calculation' strategies, called 'degree of abstractness' by Kouba (1989) and Mulligan (1992), and 'modeling' strategies by Mulligan (1992) and Carpenter et al. (1993). Combining their descriptions leads to the five categories summarised in Table 1 below. Students may model the problem situation using objects or fingers, by drawing ikons or tallies; or they may not model the situation externally at all. These strategies have been reported as occurring in conjunction with all five strategies in Table 1.

Table 1: Children's solution strategies for one-step multiplicative word problems

1	*Direct counting.* Physical materials are used to model the problem and the objects are simply counted without any obvious reference to the multiplicative structure, e.g. "1,2,3,4,5,6,7,8,9".
2	*Rhythmic counting.* Counting follows the structure of the problem, e.g. "1,2,3, 4,5,6 or 6.5.4.3.2." Simultaneously with counting, a second count is kept of the number of groups.
3	*Skip counting.* Counting is done in multiples (e.g. "3,6,9" or "9, 6,3"), making it easier to keep count of the number of groups.
4	*Additive calculation.* Counting is replaced by calculations such as "3+3=6; 6+3=9" or " 9-6=3, 6-3 =3.
5	*Multiplicative calculation.* Calculations now take the form of known facts e.g."3 times 3 is 9" or derivations from a known fact, e.g. "6 x 3 = 5 x 3 + 3".

What is an intuitive model?

The notion of intuitive models (also called implicit, tacit, or informal models) appears to have originated with Fischbein et al. (1985). They hypothesised that "each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model" (p.4) which mediates the search for the operation needed to solve a problem. They also claim that

> *...when trying to discover the nature of the intuitive model that a person tacitly associates with a certain operation, one has to consider some practical behaviour that would be the enactive, effectively performable counterpart of the operation (p.5).*

In other words, an intuitive model is an internalisation of the physical operation involved in the corresponding problem situation.

The above seems to imply a direct correspondence between intuitive model and semantic structure. Indeed, Fischbein et al. (1985) describe three intuitive models which are clearly semantic structures. For multiplication, they hypothesise a repeated addition model "in which a number of collections of the same size are put together "(p.6) -clearly the equivalent groups semantic structure. For division they describe partition and quotition models and claim that "the structure of the problem determines which model is activated" (p.7). They claimed to show that these intuitive models affected the performance of students in Grades 5, 7, and 9 as they solved problems where the multiplier was less than one. However, they report no direct evidence of students' intuitive models.

Kouba (1989) was stimulated by the Fischbein et al. (1985) study to investigate children's intuitive models by observing young children's problem-solving strategies. She used the same three semantic structures as Fischbein et al. (1985). For our purposes a most pertinent finding was that partition and quotition problems did not generate different calculation strategies; there were similarities in the intuitive models that children appeared to have for measurement and partitive division. This finding suggests that there may be in fact no direct correspondence between the semantic structure of a problem and the method which a child uses to solve it. The implication we draw from Kouba's study is that it would be valuable to examine children's solution strategies in more detail in order to infer their intuitive models. It would seem crucial when talking about a child's intuitive model, to relate it to how the child actually solves the problem rather than relying on classification based solely on semantic structure. We propose to define an intuitive model as an internal mental structure which corresponds to a class of calculation strategies.

Intuitive models of multiplication and division

Anghileri's (1989) results, obtained over six semantic structures, suggests only three intuitive models for multiplication: unitary counting, repeated addition, and multiplicative calculation. Steffe (1988) highlights the important leap from unitary counting, "three ones" to "one three", in understanding multiplication. It seems reasonable to combine rhythmic counting, skip counting and additive calculation into the one repeated addition model if we observe that all these strategies are based on the same principle of double counting. For division, Kouba's (1989) results suggest at least four intuitive models: sharing, repeated taking away, building up, and multiplicative calculation. Both repeated taking away and building up appear to be based on the same principles as repeated addition. Other classes of strategies might appear if other problem structures are included. The above interpretation naturally raises the following questions:

1. Can the proposed intuitive models be identified in new data?
2. Do any new intuitive models appear when a broader range of semantic structures is included?
3. Does the semantic structure of the problem affect children's intuitive models?
4. How do children's intuitive models change over time, especially as a result of instruction?

Method

Clinical interviews were conducted by the first author four times in two successive years when the students were in Grades 2 and 3. The interview sample comprised 60 female students at the final interview. Multiplicative problems were constructed to represent five of the ten semantic structures identified by Greer (1992) and were presented with two number sizes. (For details of problems, procedures and analysis see Mulligan (1992) and Mulligan & Mitchelmore (1995).

Results

Children's responses were first examined to find if their calculation strategies could be reliably identified. Secondly, twelve different calculation strategies were grouped to infer underlying intuitive models as shown in Table 2.

Table 2 Intuitive models for multiplication and division

	Intuitive Model	Calculation strategies
	Multiplication	
1	Direct counting	Unitary counting
2	Repeated addition	Rhythmic counting forwards
		Skip counting forwards
		Repeated adding
		Additive doubling
3	Multiplicative operation	Known multiplicative fact
		Derived multiplicative fact
	Division	
1	Direct counting	One-to-many correspondence
2.	Repeated subtraction	Unitary counting
		Sharing
		Trial-and-error grouping
		Rhythmic counting backwards
		Skip counting backwards
		Repeated subtracting
		Additive halving
3	Repeated addition	Double counting forwards
		Skip counting forwards
		Repeated adding
		Additive doubling
4	Multiplicative operation	Known multiplicative fact
		Derived multiplication fact

In summary* the direct counting model was frequently used only on the array problem and on the equivalent groups problems with large numbers. Except for the comparison problem the repeated addition model was the most frequently used model on almost all occasions. The multiplicative operation model was rare at first and began to grow at interview 3. It had become common by interview 4 when it occurred in between 24% and 65% of the correct strategies on each problem.

Direct counting was almost always more frequent for large number problems than small numbers problems; the repeated addition and multiplicative operation models were almost always less frequent for large number problems. For division, the direct counting model was mainly observed in the quotition problems and in large number partition problems. The repeated subtraction model was only consistently common on a partition small number problem. Repeated addition was common on all problems and almost always the most frequent correctly used model. The multiplicative operation model was rarely used in Grade 2 but began to appear in Grade 3, but was not as frequent as for multiplication.

Most students were not consistent in their intuitive models at any interview stage. Problem characteristics, such as semantic structure and specific numbers used seemed to influence which intuitive model would be correctly employed. At each interview, there were some students who used the same intuitive model on all problems but there were others who used as many as three different models. On the other hand, students showed a consistent progression of intuitive models from interview to interview within each problem.

Discussion

Among students in Grades 2 and 3 we have been able to clearly identify three intuitive models for multiplication (direct counting, repeated addition and multiplicative operations) and four for division (direct counting, repeated subtraction, repeated addition and multiplicative operation). We also found a clear variation in the intuitive models successfully employed in different problems. However, the structure of the preferred intuitive models did not necessarily correspond to the semantic structure of the problems: All intuitive models were employed across all problems. Many of the observed differences in preferred model were readily explained by the size of the numbers used, the particular multiples involved, and the students' relative familiarity with the situations and language used to describe them.

We did not expect to find such a strong preference for the repeated addition model of division across all semantic structures. This appears to be a result of the close connection which students see between division and multiplication problem situations before they receive instruction in division. The same close connection is evidenced by students' spontaneous use of an operation model for division shortly after instruction in multiplication.

These conclusions are limited by the problems and number combinations used in the study. A clearer picture may have been obtained if the numbers had been better controlled. Despite these limitations, our findings are in contrast to the intuitive models proposed by Fischbein et al. (1985)- models which are essentially reflections of three common semantic structures (equivalent groups, partition and quotition). We found no evidence that Grade 2 and 3 students solve problems with these three semantic structures in any consistently different manner, or that they use only strategies corresponding to these structures when solving other semantic problems. Instead it would seem that they use a small set of intuitive models which they can apply to both multiplication and division problems of all semantic structures. *The intuitive model used to solve a particular problem is not determined by the semantic structure of the problem but by the mathematical structure which the student is able to impose on it.*

Our results allow us to form a tentative picture of how young children's intuitive understanding of multiplication and division of whole numbers evolves. It would appear that they acquire a widening repertoire of increasingly efficient intuitive models which are applicable to whole number situations where the structure of each model derives from the previous one.

We might ask why the first intuitive model to develop after direct counting is repeated addition or subtraction, and why the repeated addition model is preferred across all semantic structures. The reason appears to lie in the fact that in every multiplicative situation "*there must be equal-sized groups* "(Confrey, 1994,p. 307). If one excludes multiplicative operation, the only way in which the result of a replication or magnification can be calculated is by direct counting or repeated addition. The teacher's task is to help students recognise the repeated addition structure of a variety of situations encouraging efficient calculation strategies. In particular, multiplication and division should be strongly linked to repeated addition and to each other. It does not appear to be advisable to teach students sharing or repeated subtraction as division techniques; neither of these is closely linked to multiplication.

Learning multiplication and division of rational numbers

At first glance there would seem to be little connection between whole number and rational number multiplication. As Fischbein et al. (1985) remarked,"One cannot intuitively conceive of taking a quantity 0.63 times" (p.6). However, "0.63 times something" means partition into 100 equal-sized groups and take 63 of them", so the equal-sized grouping structure of whole number multiplication is still present when dealing with rational numbers. We could therefore expect to find a close link between students' intuitive models for whole number and rational number multiplication.

The poor performance of older students on rational multiplicative problems may be explained by their lack of opportunity to develop intuitive rational number multiplication and that they may be unaware of the equal-group structure of all multiplicative situations. One practical solution would be for students to experience multiplicative problems involving rational numbers at the same time, or soon after whole number problems. For example, Confrey and Smith (1995) describe a broad category of measurement situations which appear familiar to young children and easily extend into rational numbers, but which are currently neglected in the school curriculum. Also, Behr,Harel,Post and Lesh (1994) show how rational number arithmetic can be approached in such a way to make the connection with whole numbers explicit. Only further research monitoring the growth of multiplicative reasoning using an integrated approach will reveal whether such approaches are more successful and meaningful than traditional methods.

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